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Optimal Ordering Policy for Deteriorating Items with Partial Backlogging under Permissible Delay in Payments

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Abstract. In 1985, Goyal developed an Economic order quantity (EOQ) model under conditions of permissible delay in payments. Jamal et al. then generalized Goyal's model for deteriorating items with completely backlogging. However, they only ran several simulations to indicate that the total relevant cost may be convex. Recently, Teng amended Goyal's model by considering the difference between unit price and unit cost, and provided an alternative conclusion that it makes economic sense for some retailers to order less quantity and take the benefits of the permissible delay more frequently. However, he did not consider deteriorating items and partial backlogging. In this paper, we establish a general EOQ model for deteriorating items when the supplier offers a permissible delay in payments. For generality, our model allows not only the partial backlogging rate to be related to the waiting time but also the unit selling price to be larger than the unit purchase cost. Consequently, the proposed model includes numerous previous models as special cases. In addition, we mathematically prove that the total relevant cost is strictly pseudo-convex so that the optimal solution exists and is unique. Finally, our computational results reveal six managerial phenomena.

Key words: deteriorating items, finance, inventory, partial backlogging

1. Introduction

The traditional economic order quantity (EOQ) model assumes that the retailer must pay for the items as soon as the items are received. Indeed, goods are seldom paid for immediately after they appear in a retailer's stockroom. In market behaviors, nearly all firms rely to some extent on trade credit as a source of short-term funds. In fact, small firms generally use trade credit more extensively than large firms. When monetary policy is tight and credit is difficult to obtain, small firms tend to increase their reliance on trade credit. That is, during periods of tight money, small firms that are unable to obtain sufficient funds through normal channels may obtain financing indirectly from large suppliers by "stretching" their payment periods and extending accounts payable. Large firms often are willing to finance their smaller customers in this manner in order to preserve their markets. Ordinarily the forms of trade credit are open account, promissory note, and trade acceptance (e.g., see Solomon and Pringle, 1980). As to a retailer conducting business with foreign suppliers, it must pay attention to the exchange rates between foreign currencies and its own currency, and the effects of fluctuating currency values in its financial analysis.

In this paper, we assume that a supplier often offers his retailers a period of time, perhaps 30 days, to settle the amount owed to him. Usually, there is no interest charge if the outstanding amount is paid within the permissible delay period of 30 days. Note that this credit term in financial management is denoted as "net 30" (e.g., see Brigham, 1995). However, if the payment is not paid in full by the end of the permissible delay period, then interest is charged on the outstanding amount under the terms and conditions agreed upon. Therefore, a retailer will earn the interest on the accumulated revenue received, and delay payment until the last moment of the permissible period allowed by the supplier. The permissible delay in payments reduces the cost of holding inventory to the retailer for the duration of the permissible period. Hence, it is a marketing strategy for the supplier to attract new retailers who consider it to be a type of cost (or price) reduction. However, the strategy of granting credit terms adds not only an additional cost but also an additional dimension of default risk to the supplier.

Goyal (1985) developed an EOQ model under conditions of permissible delay in payments. He assumed that the unit purchase cost is the same as the selling price per unit, and concluded that "as a result of permissible delay in setting the replenishment account, the economic replenishment interval and order quantity generally increases marginally, although the annual cost decreases considerably." Although Dave (1985) amended Goyal's model by the fact that the selling price is necessarily higher than its purchase price, his viewpoint did not draw much attention in subsequent research. Aggarwal and Jaggi (1995) then extended Goyal's model to include deteriorating items. Jamal et al. (1997) further generalized the model to allow for shortages and deterioration. However, they only ran several simulations to indicate that the total relevant cost may be convex. Hwang and Shinn (1997) developed the optimal pricing and lot sizing for the retailer under the condition of permissible delay in payments. Liao et al. (2000) developed an inventory model for a stock-dependent demand rate when a delay in payment is permissible. Recently, Chang and Dye (2001) extended the model by Jamal et al. to allow for not only a varying deterioration rate but also requiring the backlogging rate to be inversely proportional to the waiting time. All of the above models (except Dave 1985) ignored the difference between unit price and unit cost, and obtained the same conclusion as in Goyal (1985). In contrast, Jamal et al. (2000) and Sarker et al. (2000) amended Goyal's model by considering the difference between unit price and unit cost, and concluded from computational results that the retailer should settle his account relatively sooner as the unit selling price increases relative to the unit cost. Recently, Teng (2002) provided an alternative conclusion from Goyal (1985), and mathematically proved that it makes economic sense for a well-established buyer to order less quantity and take the benefits of the permissible delay more frequently. However, he did not include deteriorating items and partial backlogging. Chang et al. (2003) then extended Teng's model, and established an EOQ model for deteriorating items in which the supplier provides a permissible delay to the retailer if the order quantity is greater than or equal to a predetermined quantity.

For fashionable commodities, trendy apparel, and high-tech products with short product life cycle, the willingness for a customer to wait for backlogging during a shortage period is diminishing with the length of the waiting time. Hence, the longer the waiting time is, the smaller the backlogging rate would be. To reflect this phenomenon, Abad (1996) proposed several distinct backlogging rates to be decreasing functions of waiting time. Chang and Dye (1999) then developed a finite-horizon inventory model by using Abad's reciprocal backlogging rate. Concurrently, Papachristos and Skouri (2000) established a multi-period inventory model based on Abad's exponential backlogging rate. Recently, Teng et al. (2002) extended the fraction of unsatisfied demand backordered to any decreasing function of the waiting time up to the next replenishment.

In this paper, we establish an appropriate and general EOQ model for a retailer to determine its optimal shortage interval and replenishment cycle when the supplier offers a permissible delay in payments. For generality, our model allows not only the partial backlogging rate to be related to the waiting time but also the unit selling price to be larger than the unit purchase cost. Consequently, the proposed model is in a general framework that includes numerous special cases presented in Aggarwal and Jaggi (1995), Chang and Dye (1999, 2001), Dave (1985), Goyal (1985), Jamal et al. (1997), Papachristos and Skouri (2000), Teng (2002) and Teng et al. (1999, 2002, 2003). We then mathematically prove that the total relevant cost (i.e., the sum of ordering cost, purchase cost, backlogging cost, cost of lost sales, interest payable, and interest earned) is a strictly pseudo-convex function. As a result, there exists a unique optimal solution to our proposed model. In contrast to our theoretical result, Jamal et al. (1997) only ran several simulations that indicated the surface of the total relevant cost may be convex. Finally, we perform several sensitivity analyses and obtain six managerial phenomena.

2. Assumptions and Notation

The mathematical models proposed in this paper are based on the following assumptions:

(1) The demand for the item is constant with time.
(2) Lead time is zero.
(3) The initial inventory level is zero.
(4) Shortages are allowed. However, the longer the waiting time, the smaller the backlogging rate. Hence, we assume that the fraction of shortages backordered $\beta(x)$ (with $0 < \beta(x) \leqslant 1$ and $\beta(0) = 1$) is a decreasing and differentiable function of x, where x is the waiting time up to the next replenishment. Consequently, the first derivative of $\beta(x)$, $\beta'(x)$, is less than or equal to zero, for all $x \geqslant 0$.
(5) During the shortage period, the retailer can obtain the interest earned on the order cost due to the delay in placing the order. However, the retailer will pay not only the shortage cost for backlogged items but also the opportunity cost for lost sales. Note that the cost of lost sales in a minimization problem is the sum of the cost of lost revenue and the cost of lost goodwill (e.g., see Teng et al. 2002). As a result, the cost of lost sales must be greater than or equal to the selling price.
(6) A constant fraction of the on-hand inventory deteriorates per unit of time and there is no repair or replacement of the deteriorated inventory.
(7) The supplier provides a certain fixed credit period to settle the account. During the trade credit period, sales revenue is deposited in an interest bearing account. At the end of the permissible delay, the retailer has two ways to pay off the loan. One is that the retailer pays off all units sold, keeps the rest for the use of the other activities, and starts paying for the interest charges on the items in stocks. The other is that the retailer pays off the loan whenever he/she has money, such as in Jamal et al. (2000), Sarker et al. (2000), and Chang and Teng (2004). In this paper, we will discuss both possible ways.

In addition, the following notation is used throughout this paper.

C_0 the ordering cost per order
C_h the holding cost excluding interest charge (\$/unit/per year)
C_s the shortage cost for backlogged items (\$/unit/per year)
p the selling price per unit
C_l the unit cost of lost sales, which is the sum of lost revenue and lost goodwill, hence $C_\mathrm{l} \geqslant p$ (for detail, see Teng et al. 2002)
C_p the unit purchasing cost, with $C_\mathrm{p} < p$

D	demand rate (units/per year)
I_e	the annual interest rate earned per dollar
I_c	the annual interest rate charged per dollar
M	the period of permissible delay in settling the account, in years
S	the length of the shortage period, in years
T	the length of the inventory cycle, $0 \leqslant S < T$, in years
θ	the constant deterioration rate, where $0 \leqslant \theta < 1$
Q	the order quantity
$I_1(t)$	the level of negative inventory at time t, $0 \leqslant t \leqslant S$
$I_2(t)$	the level of positive inventory at time t, $S \leqslant t \leqslant T$
$\mathrm{AC}(S, T)$	the total annual relevant cost, which is a function of S and T
$\mathrm{AC}_1\,(S, T)$	the total annual relevant cost for Case 1, in which the retailer pays off only units sold, and keeps the profit for the use of the other activities at time $M + S$ with $S < M + S \leqslant T$
$\mathrm{AC}_2\,(S, T)$	the total annual relevant cost for Case 2, in which the retailer pays off only units sold, and keeps the profit for the use of the other activities at time $M + S$ with $S < T \leqslant M + S$
$\mathrm{AC}_3\,(S, T)$	the total annual relevant cost for Case 3, in which the retailer pays off the total amount on its account at time $M + S$ with $S < M + S \leqslant T$
$\mathrm{AC}_4\,(S, T)$	the total annual relevant cost for Case 4, in which the retailer pays off the total amount on its account at time $M + S$ with $S < T \leqslant M + S$

Next, we will discuss seven possible relevant costs related to the problem, and how each of the costs will weigh into the difficulty of determining whether the total cost function is convex.

3. Mathematical Model

The total relevant cost per cycle consists of: (a) cost of placing an order, (b) cost of shortage, (c) cost of lost sales, (d) cost of purchasing, (e) cost of holding inventory (excluding interest charges), (f) cost of interest charges for unsold items after the permissible delay M (note that this cost occurs only if $S < M + S \leqslant T$), and (g) interest earned from sales revenue during the permissible period. However, there are two possible total relevant costs based on the values of T and $M + S$.

The supplier offers the retailer the permissible payment delay, M, in order to stimulate the demand. Consequently, there are two possible scenarios:

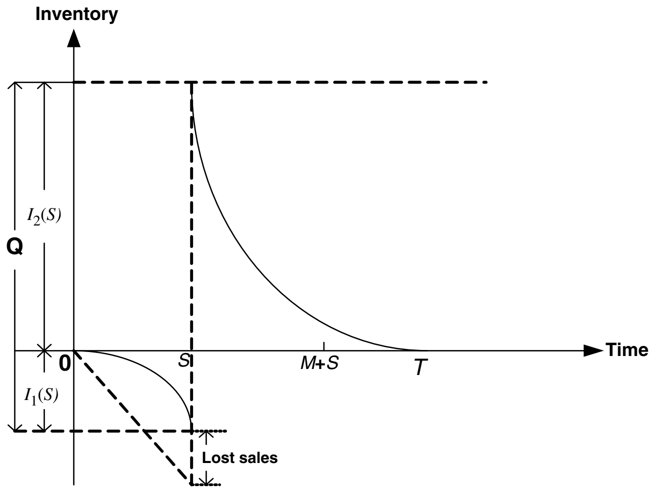

Figure 1. Inventory system with $S < M + S \leqslant T$.

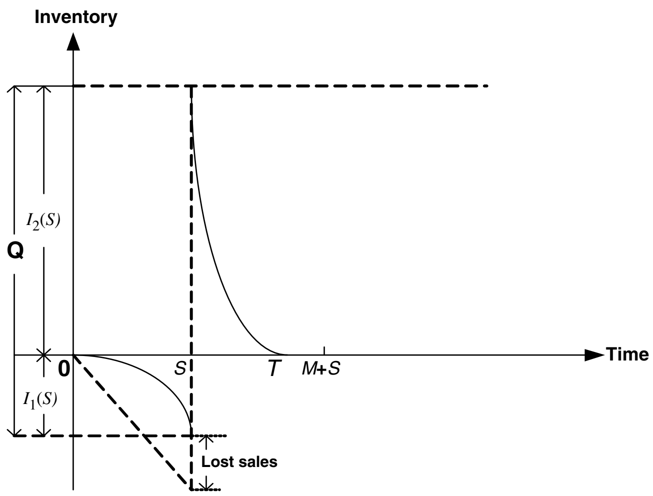

Figure 2. Inventory system with $S < T \leqslant M + S$.

(A) The end of the credit period, $M + S$, is shorter than or equal to the length of the replenishment cycle T (i.e., $S < M + S \leqslant T$). The graphical representation is depicted in Figure 1.
(B) The end of the credit period, $M + S$, is longer than or equal to the length of the replenishment cycle T (i.e., $S < T \leqslant M + S$), The graphical representation in this case is depicted in Figure 2.

From Figures 1 and 2, we know that during the time $t \in [0, S]$ the level of negative inventory $I_1(t)$ is the cumulative backlogged demand up to t. At time S, an order is made and the quantity received is used partly to meet the accumulated backlogged demand in [0, S]. The positive inventory $I_2(t)$ during $[S, T]$ gradually decreases due to consumption and deterioration. Hence, the variation of inventory with respect to time can be described by the following differential equations:

$$\frac{\mathrm{d}I_1(t)}{\mathrm{d}t} = -\beta\,(S-t)\,D, \quad 0 \leqslant t \leqslant S, \tag{1}$$

$$\frac{\mathrm{d}I_2(t)}{\mathrm{d}t} + \theta I_2\,(t) = -D, \quad S \leqslant t \leqslant T, \tag{2}$$

with the boundary conditions: $I_1\,(0) = I_2(T) = 0$. Consequently, the solutions to Equations (1) and (2) are given by

$$I_1\,(t) = -\int_0^t \beta(S-u)\,D\ \mathrm{d}u, \quad 0 \leqslant t \leqslant S, \tag{3}$$

and

$$I_2\,(t) = \frac{D}{\theta}[\mathrm{e}^{\theta(T-t)} - 1], \quad S \leqslant t \leqslant T, \tag{4}$$

respectively.

Therefore, the total relevant cost per replenishment cycle consists of the following elements.

(a) The ordering cost per order is fixed at C_0. However, the retailer earns the interest of $C_0 S I_\mathrm{e}$ on the order cost due to the delay S in placing the order. Consequently, the total ordering cost is

$$\mathrm{OC} = C_0 - C_0 S I_\mathrm{e}, \tag{5}$$

which is a linear function of S.

(b) The shortage cost for backlogged items is given by

$$\begin{aligned} SC &= C_\mathrm{s}\int_0^S [-I_1(t)]\mathrm{d}t = C_\mathrm{s}\int_0^S\int_0^t \beta(S-u)\,D\ \mathrm{d}u\ \mathrm{d}t \\ &= C_\mathrm{s}D\int_0^S (S-u)\,\beta(S-u)\,\mathrm{d}u = C_\mathrm{s}D\int_0^S t\beta\,(t)\,\mathrm{d}t, \end{aligned} \tag{6}$$

which is a convex function of S because $\mathrm{d}^2SC/\mathrm{d}S^2 = C_\mathrm{s}D[\beta(S) - S\beta'(S)] > 0$.

(c) The opportunity cost of lost sales is given by

$$\begin{aligned} \mathrm{LS} &= C_\mathrm{l}\int_0^S [1-\beta(S-u)]\,D\,\mathrm{d}u = C_\mathrm{l}D\left[S - \int_0^S \beta\,(S-u)\,\mathrm{d}u\right] \\ &= C_\mathrm{l}D\left[S - \int_0^S \beta(t)\mathrm{d}t\right], \end{aligned} \tag{7}$$

which is a convex function of S because $\mathrm{d}^2 LS/\mathrm{d}S^2 = -C_l D\beta'(S) > 0$.

(d) The purchasing cost can be obtained by

$$\begin{aligned} PC &= C_\mathrm{p} Q = C_\mathrm{p}\left[I_2(S) - I_1(S)\right] \\ &= C_\mathrm{p}\left[\frac{D}{\theta}(\mathrm{e}^{\theta(T-S)} - 1) + \int_0^S \beta\,(S-u)\,D\,\mathrm{d}u\right] \\ &= C_\mathrm{p}\left[\frac{D}{\theta}(\mathrm{e}^{\theta(T-S)} - 1) + \int_0^S \beta\,(t)\,D\,\mathrm{d}t\right], \end{aligned} \tag{8}$$

which is neither convex nor concave because the Hessian matrix is

$$\left(\frac{\partial^2 PC}{\partial T^2}\right)\left(\frac{\partial^2 PC}{\partial S^2}\right) - \left(\frac{\partial^2 PC}{\partial T \partial S}\right)^2 = C_\mathrm{p} D\theta \mathrm{e}^{\theta(T-S)}\beta'(S) < 0.$$

(e) The holding cost during the interval $[S, T]$ is given by

$$\begin{aligned} HC &= C_\mathrm{h}\int_S^T I_2(t)\mathrm{d}t = C_\mathrm{h}\int_S^T \frac{D}{\theta}(\mathrm{e}^{\theta(T-t)} - 1)\mathrm{d}t \\ &= \frac{C_\mathrm{h} D}{\theta^2}\left[\mathrm{e}^{\theta(T-S)} - 1 - \theta\,(T-S)\right]. \end{aligned} \tag{9}$$

We cannot determine whether HC is convex or not because the Hessian matrix is

$$\left(\frac{\partial^2 HC}{\partial T^2}\right)\left(\frac{\partial^2 HC}{\partial S^2}\right) - \left(\frac{\partial^2 HC}{\partial T \partial S}\right)^2 = 0.$$

Regarding interests charged and earned (i.e., costs of (f) and (g)), we have the following four possible cases based on how the retailer pays off the loan, and the values of T, and $S + M$. These four cases are described below.

Case 1. The retailer pays off only units sold, and keeps the profit for the use of the other activities at time $M + S$ with $S < M + S \leqslant T$.

In this case, the end point of credit period $M + S$ is shorter than or equal to the period of the replenishment cycle T. As a result, the products unsold after $M + S$ must be financed with an annual rate I_c. Therefore, the interest charged per cycle is given by

$$IC = C_\mathrm{p} I_\mathrm{c}\int_{M+S}^T I_2(t)\mathrm{d}t = \frac{C_\mathrm{p} I_\mathrm{c} D}{\theta^2}[\mathrm{e}^{\theta(T-M-S)} - 1 - \theta\,(T-M-S)]. \tag{10}$$

We cannot determine whether IC is convex in (S, T) because the Hessian matrix is

$$\left(\frac{\partial^2 IC}{\partial T^2}\right)\left(\frac{\partial^2 IC}{\partial S^2}\right)-\left(\frac{\partial^2 IC}{\partial T\partial S}\right)^2=0.$$

On the other hand, during the period $[S, M+S]$, the retailer deposits the sales revenue to earn interest with an annual rate I_e. Hence, the interest earned per cycle is given by

$$\begin{aligned} IE &= pI_eM\int_0^S \beta(S-u)D\,\mathrm{d}u + pI_e\int_S^{M+S} D\cdot(u-S)\mathrm{d}u, \\ &= \frac{pDI_eM}{2}\left[M+2\int_0^S \beta(S-u)\mathrm{d}u\right] \\ &= \frac{pDI_eM}{2}\left[M+2\int_0^S \beta(t)\mathrm{d}t\right], \end{aligned} \tag{11}$$

which is a concave function of S because $\mathrm{d}^2IE/\mathrm{d}S^2 = pDI_eM\beta'(S) < 0$.

Therefore, the total annual relevant cost is calculated as

$$\begin{aligned} \mathrm{AC}_1(S,T) &= \frac{OC+HC+SC+LS+PC+IC-IE}{T} \\ &= \frac{C_0(1-SI_e)}{T} + \frac{C_hD\left[e^{\theta(T-S)}-1-\theta(T-S)\right]}{\theta^2 T} \\ &\quad + \frac{C_sD\int_0^S t\beta(t)\mathrm{d}t}{T} + \frac{C_lD\left[S-\int_0^S \beta(t)\mathrm{d}t\right]}{T} \\ &\quad + \frac{C_pD\left[(e^{\theta(T-S)}-1)+\theta\int_0^S \beta(t)\mathrm{d}t\right]}{\theta T} \\ &\quad + \frac{C_pDI_c\left[e^{\theta(T-M-S)}-1-\theta(T-M-S)\right]}{\theta^2 T} \\ &\quad - \frac{pDI_eM\left[M+2\int_0^S \beta(t)\mathrm{d}t\right]}{2T}. \end{aligned} \tag{12}$$

Note that the denominator of $\mathrm{AC}_1(S,T)$ is the decision variable T, not a constant. Consequently, it is difficult to determine whether $\mathrm{AC}_1(S,T)$ is convex or not. However, we are able to prove that $\mathrm{AC}_1(S,T)$ is strictly pseudo-convex in the next section.

Case 2. The retailer pays off only units sold, and keeps the profit for the use of the other activities at time $M+S$ with $S < T \leqslant M+S$.

In this case, the end point of credit period $M+S$ is longer than or equal to the period of the replenishment cycle T. The ordering cost, shortage cost, opportunity cost of lost sales, purchasing cost and holding cost are completely analogous with those in Case 1. However, in this case, the retailer has no interest to pay while it earns the interest during the credit period. Therefore, the interest earned in this case is given by

$$\begin{aligned} IE &= pDI_{\mathrm{e}}\left[\frac{(T-S)^2}{2}+(T-S)\,(M+S-T)+M\int_0^S \beta(S-u)\mathrm{d}u\right] \\ &= pDI_{\mathrm{e}}\left[\frac{(T-S)^2}{2}+(T-S)\,(M+S-T)+M\int_0^S \beta(t)\mathrm{d}t\right], \end{aligned} \tag{13}$$

which is a concave function of (S, T) because the Hessian matrix is positive definite,

$$\left(\frac{\partial^2 IE}{\partial T^2}\right)\left(\frac{\partial^2 IE}{\partial S^2}\right)-\left(\frac{\partial^2 IE}{\partial T\partial S}\right)^2=-pDI_{\mathrm{e}}M\beta'(S)>0,$$

and

$$\frac{\partial^2 IE}{\partial T^2}=-pDI_{\mathrm{e}}<0.$$

As a result, the total annual relevant cost is given as

$$\begin{aligned} \mathrm{AC}_2\,(S,T) &= \frac{OC+HC+SC+LS+PC-IE}{T} \\ &= \frac{C_0\,(1-SI_{\mathrm{e}})}{T}+\frac{C_hD\left[\mathrm{e}^{\theta(T-S)}-1-\theta\,(T-S)\right]}{\theta^2T} \\ &\quad +\frac{C_{\mathrm{s}}D\int_0^S t\beta(t)\mathrm{d}t}{T}+\frac{C_{\mathrm{l}}D[S-\int_0^S \beta(t)\mathrm{d}t]}{T} \\ &\quad +\frac{C_{\mathrm{p}}D[(\mathrm{e}^{\theta(T-S)}-1)+\theta\int_0^S \beta(t)\mathrm{d}t]}{\theta T} \\ &\quad -\frac{pDI_{\mathrm{e}}[(T-S)^2+2\,(T-S)\,(M+S-T)+2M\int_0^S \beta(t)\mathrm{d}t]}{2T}. \end{aligned} \tag{14}$$

Again, whether $\mathrm{AC}_2\,(S,T)$ is convex is hard to be determined because its denominator is T, not a constant. However, we are able to show that it is strictly pseudo-convex in the next section.

Case 3. The retailer pays off the total amount on its account at time $M+S$ with $S<M+S\leqslant T$.

During $[S, M+S]$ period, the retailer deposits the revenue from sales into an account that earns I_e per dollar per unit time. Therefore, the accumulative interest earned per cycle at time $M+S$ is

$$pI_e\left[M\int_0^S \beta(s-t)D\,\mathrm{d}t+\int_0^M D\,t\mathrm{d}t\right]=pI_eD\left(M\int_0^S \beta(t)\mathrm{d}t+M^2/2\right) \tag{15}$$

Hence, the retailer has $pD(\int_0^S \beta(t)\mathrm{d}t+M)+pI_eD(M\int_0^S \beta(t)\mathrm{d}t+M^2/2)$ in the account at time $M+S$. In the meantime, at time $M+S$, the retailer owes the wholesaler the total purchase cost

$$C_pQ=C_p\left[\frac{D}{\theta}\left(\mathrm{e}^{\theta(T-S)}-1\right)+\int_0^S \beta(t)D\,\mathrm{d}t\right]. \tag{16}$$

From the difference between the total purchase cost and the money in the account at time $M+S$, we have the following two possible cases:

$$\text{(a)}\quad pD\left(\int_0^S \beta(t)\mathrm{d}t+M\right)+pI_eDM\left(\int_0^S \beta(t)\mathrm{d}t+\frac{M}{2}\right) \geqslant C_p\left[\frac{D}{\theta}(\mathrm{e}^{\theta(T-S)}-1)+\int_0^S \beta(t)D\,\mathrm{d}t\right],$$

and

$$\text{(b)}\quad pD\left(\int_0^S \beta(t)\mathrm{d}t+M\right)+pI_eDM\left(\int_0^S \beta(t)\mathrm{d}t+\frac{M}{2}\right) \leqslant C_p\left[\frac{D}{\theta}(\mathrm{e}^{\theta(T-S)}-1)+\int_0^S \beta(t)D\,\mathrm{d}t\right].$$

For simplicity, we will discuss only the first case. The reader can easily obtain the similar results for the other case.

If $pD(\int_0^S \beta(t)\mathrm{d}t+M)+pI_eDM(\int_0^S \beta(t)\mathrm{d}t+\frac{M}{2})\geqslant C_p[\frac{D}{\theta}(\mathrm{e}^{\theta(T-S)}-1)+\int_0^S \beta(t)D\ \mathrm{d}t]$, then the retailer is able to pay off the total purchase cost at time M+S. Consequently, there is no interest payable to the retailer in this case (i.e., $IC=0$). In addition, the retailer receives the interest earned during $[M+S,T]$ as

$$I_e(T-M-S)f(S,T)+pDI_e(T-M-S)^2/2, \tag{17}$$

where,

$$f(S,T)=pD\left(\int_0^S \beta(t)\mathrm{d}t+M\right)+pDI_{\mathrm{e}}M\left(\int_0^S \beta(t)\mathrm{d}t+M/2\right)$$
$$-C_{\mathrm{p}}D\left[\frac{1}{\theta}\left(\mathrm{e}^{\theta(T-S)}-1\right)+\int_0^S \beta(t)\mathrm{d}t\right]. \tag{18}$$

As a result, the retailer receives the interest earned during $[0,T]$ as

$$IE=pDI_{\mathrm{e}}\left[M\int_0^S \beta(t)\mathrm{d}t+\frac{M^2}{2}\right]+I_{\mathrm{e}}\,(T-M-S)\,f(S,T)$$
$$+\frac{pDI_{\mathrm{e}}(T-M-S)^2}{2}. \tag{19}$$

Therefore, the total annual relevant cost is calculated as

$$AC_3\,(S,T)=\frac{OC+HC+SC+LS+PC-IE}{T}$$
$$=\frac{C_0\,(1-SI_{\mathrm{e}})}{T}+\frac{C_{\mathrm{h}}D\left[\mathrm{e}^{\theta(T-S)}-1-\theta\,(T-S)\right]}{\theta^2 T}$$
$$+\frac{C_{\mathrm{s}}D\int_0^S t\beta(t)dt}{T}+\frac{C_{\mathrm{l}}D[S-\int_0^S \beta(t)\mathrm{d}t]}{T}$$
$$+\frac{C_{\mathrm{p}}D[\left(\mathrm{e}^{\theta(T-S)}-1\right)+\theta\int_0^S \beta(t)\mathrm{d}t]}{\theta T}$$
$$-\frac{pDI_{\mathrm{e}}M\left(2\int_0^S \beta(t)dt+M\right)}{2T}$$
$$-\frac{2I_{\mathrm{e}}\,(T-M-S)\,f(S,T)+pDI_{\mathrm{e}}\,(T-M-S)^2}{2T}. \tag{20}$$

Due to its complexity, we could not determine whether $AC_3\,(S,T)$ is strictly pseudo-convex or not.

Case 4. The retailer pays off the total amount on its account at time $M+S$ with $S<T\leqslant M+S$.

Since $T\leqslant M+S$, the retailer will pay off the total purchase cost at time $M+S$ for either Case 2 and Case 4. Consequently, Case 4 is exactly the same as Case 2.

4. Theoretical Results and an Algorithm

In this section, we will show that the total annual relevant cost in each of first two cases is strictly pseudo-convex. As a result, the optimal solution to the problem exists and is unique. Finally, we propose an algorithm to solve the problem. Likewise, the reader can obtain similar results for the other two cases.

Case 1. The retailer pays off only units sold, and keeps the profit for the use of the other activities at time $M+S$ with $S<M+S\leqslant T$.

To minimize the total annual relevant cost, taking the first derivative of $\mathrm{AC}_1\,(S,T)$ with respect to S and T, and setting the results to be zero, we obtain

$$\begin{aligned}\frac{\partial \mathrm{AC}_1(S,T)}{\partial T} &= \frac{-AC_1\,(S,T)}{T}\\ &\quad+\frac{D}{T}\left[\mathrm{e}^{\theta(T-S)}\left(\frac{C_\mathrm{h}}{\theta}+C_\mathrm{p}+\frac{C_\mathrm{p}I_\mathrm{c}\mathrm{e}^{-\theta M}}{\theta}\right)-\frac{C_\mathrm{h}}{\theta}-\frac{C_\mathrm{p}I_\mathrm{c}}{\theta}\right]\\ &=0,\end{aligned} \tag{21}$$

and

$$\begin{aligned}\frac{\partial \mathrm{AC}_1(S,T)}{\partial S} = \frac{D}{T}\Bigg\{ &-\frac{C_0I_\mathrm{e}}{D}+\frac{C_\mathrm{h}}{\theta}\left[1-\mathrm{e}^{\theta(T-S)}\right]+C_\mathrm{s}S\beta(S)+C_\mathrm{l}[1-\beta(S)]\\ &+C_\mathrm{p}\left[-\mathrm{e}^{\theta(T-S)}+\beta(S)\right]\\ &+\frac{C_\mathrm{p}I_\mathrm{c}}{\theta}\left[1-\mathrm{e}^{\theta(T-M-S)}\right]-pI_\mathrm{e}M\beta(S)\Bigg\}=0.\end{aligned} \tag{22}$$

Let S_1^* and T_1^* denote the optimal values of S and T, respectively, then the optimal solution (S_1^*,T_1^*) satisfies (21) and (22) simultaneously. Next, we can easily prove that the associated Hessian matrix at point (S_1^*,T_1^*) is a positive definite matrix (see Appendix A for the proof). In addition, from (12) and (22), we can obtain the following theoretical results.

THEOREM 1.

(a) $\mathrm{AC}_1(S,T)$ is a strictly pseudo-convex function on the set $J=\{(S,T)\,|\,0<S<T\}$.
(b) If $C_\mathrm{s}T\beta(T)+(C_\mathrm{l}-C_\mathrm{p})[1-\beta(T)]+\frac{C_\mathrm{p}I_\mathrm{c}}{\theta}[1-\mathrm{e}^{-\theta M}]>(C_0I_\mathrm{e})/(D)+pI_\mathrm{e}M\beta(T)$, then there exists a unique interior optimal solution S_1^* (with $0<S_1^*<T_1$) to (22).
(c) If $C_\mathrm{s}T\beta(T)+(C_\mathrm{l}-C_\mathrm{p})[1-\beta(T)]+\frac{C_\mathrm{p}I_\mathrm{c}}{\theta}[1-\mathrm{e}^{-\theta M}]\leqslant(C_0I_\mathrm{e})/(D)+pI_\mathrm{e}M\beta(T)$, then the optimal solution to (22) is a boundary solution (with $S_1^*=T_1$), and the problem reduces to determine T_1^* only (Note that this case is unrealistic.)
(d) There always exists a unique interior solution T_1^* (with $0<T_1^*<\infty$) to (21).

Proof. See Appendix B.

From (21) to (22), we can obtain the following two equations:

$$T = S + \frac{1}{\theta}\ln\left[\frac{Dg_2(S)}{\theta k_1}\right], \tag{23}$$

and

$$T = \frac{1}{\theta} + \frac{g_1(S)}{Dg_2(S)}, \tag{24}$$

where $k_1 = \frac{(C_\mathrm{h}+\theta C_\mathrm{p}+C_\mathrm{p}I_\mathrm{c}\mathrm{e}^{-\theta M})D}{\theta^2}$,

$$\begin{aligned} g_1(S) = D\Bigg\{ &\frac{C_0\,(1 - SI_\mathrm{e})}{D} - \frac{C_\mathrm{h}}{\theta^2} + \frac{C_\mathrm{h}S}{\theta} + C_\mathrm{s}\int_0^S t\beta(t)\mathrm{d}t \\ &+ C_1\left(S - \int_0^S \beta(t)\mathrm{d}t\right) - \frac{C_\mathrm{p}}{\theta} + C_\mathrm{p}\int_0^S \beta(t)\mathrm{d}t - \frac{C_\mathrm{p}I_\mathrm{c}}{\theta^2} \\ &+ \frac{C_\mathrm{p}I_\mathrm{c}\,(M+S)}{\theta} - pI_\mathrm{e}\frac{M}{2}\left(M + 2\int_0^S \beta(t)\mathrm{d}t\right)\Bigg\}, \end{aligned}$$

and

$$\begin{aligned} g_2(S) = &\frac{C_\mathrm{h}}{\theta} + C_1[1-\beta(S)] + C_\mathrm{s}S\beta(S) + C_\mathrm{p}\beta(S) \\ &+ \frac{C_\mathrm{p}I_\mathrm{c}}{\theta} - \frac{C_0 I_\mathrm{e}}{D} - pI_\mathrm{e}M\beta(S). \end{aligned}$$

It is trivial from (23) and (24) that

$$S + \frac{1}{\theta}\ln\left[\frac{Dg_2(S)}{\theta k_1}\right] - \frac{1}{\theta} - \frac{g_1(S)}{Dg_2(S)} = 0. \tag{25}$$

Consequently, we can obtain the value of S_1^* from (25) by using Newton–Raphson Method (or any bisection method). We then get T_1^* by using (23) or (24). Therefore, we have the corresponding minimum total annual relevant cost $AC_1(S_1^*, T_1^*)$, which can be obtained by (12). □

Case 2. $S < T \leqslant M + S$ The case is the same as Case 4.

To minimize the total annual relevant cost, taking the first derivative of $\mathrm{AC}_2\,(S,T)$ with respect to S and T, and setting the result to be zero, we obtain

$$\frac{\partial \mathrm{AC}_2(S,T)}{\partial T} = \frac{-\mathrm{AC}_2\,(S,T)}{T}$$

$$+\frac{D}{T}\left[e^{\theta(T-S)}\left(\frac{C_h}{\theta}+C_p\right)-\frac{C_h}{\theta}-pI_e(M+S-T)\right]$$
$$=0. \tag{26}$$

$$\frac{\partial AC_2(S,T)}{\partial S}=\frac{D}{T}\left\{-\frac{C_0I_e}{D}+\frac{C_h}{\theta}\left[1-e^{\theta(T-S)}\right]+C_sS\beta(S)+C_l[1-\beta(S)]\right.$$
$$+C_p\left[-e^{\theta(T-S)}+\beta(S)\right]$$
$$\left.+pI_e[(M+S-T)-M\beta(S)]\right\}=0. \tag{27}$$

Let S_2^* and T_2^* denote the optimal values of S and T, respectively, then the optimal solution (S_2^*, T_2^*) satisfies (26) and (27) simultaneously. Again, the associated Hessian matrix at point $\left(S_2^*, T_2^*\right)$ is a positive definite matrix (The proof is similar to that in Appendix A). Likewise, from (14) and (27), we can obtain the following theoretical results.

THEOREM 2.

(a) $\mathrm{AC}_2(S,T)$ is a strictly pseudo-convex function on the set $J=\{(S,T)\,|\,0<S<T\}$.
(b) If $C_sT\beta(T)+(C_l-C_p)[1-\beta(T)]+pI_eM[1-\beta(T)]>\frac{C_0I_e}{D}$, then there exists a unique interior optimal solution S_2^* (with $0<S_2^*<T_2$) to (27).
(c) If $C_sT\beta(T)+(C_l-C_p)[1-\beta(T)]+pI_eM[1-\beta(T)]\leqslant\frac{C_0I_e}{D}$, then the optimal solution to (27) is a boundary solution (with $S_2^*=T_2$), and the problem reduces to determine T_2^* only (Note that this case does not happen in the real world.)
(d) There always exists a unique interior solution T_2^* (with $0<T_2^*<\infty$ to (26).

Proof. The proof is similar to that in Theorem 1.

From (26) to (27), we see that (S_2^*, T_2^*) satisfies the following equations simultaneously:

$$e^{\theta(T-S)}\left(\frac{C_h+\theta C_p}{\theta^2}\right)+g_3(S)=Te^{\theta(T-S)}\left(\frac{C_h+\theta C_p}{\theta}\right)+\frac{pI_eT^2}{2}. \tag{28}$$

and

$$e^{\theta(T-S)}\left(\frac{C_h+\theta C_p}{\theta}\right)+pI_eT=g_4(S), \tag{29}$$

where

$$g_3(S)=S\left(\frac{C_h}{\theta}+C_l+pI_eM-\frac{C_0I_e}{D}\right)+\frac{pI_eS^2}{2}+C_S\int_0^S t\beta(t)\mathrm{d}t$$

$$-C_1\int_0^S \beta(t)\mathrm{d}t + C_p\int_0^S \beta(t)\mathrm{d}t$$
$$-pI_eM\int_0^S \beta(t)\mathrm{d}t + \frac{C_0}{D} - \frac{C_h+\theta C_p}{\theta^2}.$$

and

$$g_4(S)=\beta(S)\left(C_S S - C_1 + C_p - pI_eM\right) - \frac{C_0 I_e}{D} + C_1 + pI_e\,(M+S) + \frac{C_h}{\theta}.$$

Solving (28) and (29) by Newton–Raphson method (or any other bisection method), we can obtain the values of S_2^* and T_2^*. Hence, the corresponding minimum total annual relevant cost $AC_2(S_2^*, T_2^*)$ can be obtained by (14). Combining Cases 1 and 2, and assuming S^* to be an interior solution for reality, we propose the following algorithm to obtain the optimal solution of (S, T). From Theorems 1 and 2, the reader can easily develop a similar algorithm as below when S^* is a boundary solution. □

ALGORITHM

Step 1. Solving (17) and (18), we get the interior solution of (S, T), denoted by (S_1, T_1). If $S_1 + M \leqslant T_1$, then (S_1, T_1) is a solution to Case 1, and the corresponding $AC_1(S_1, T_1)$ can be obtained by (12). Otherwise, (S_1, T_1) is infeasible and set $AC_1(S_1, T_1) = \infty$.

Step 2. Solving (22) and (23), we get the interior solution of (S, T), denoted by (S_2, T_2). If $T_2 \leqslant S_2 + M$, then (S_2, T_2) is a solution to Case 2, and the corresponding $AC_2(S_2, T_2)$ can be obtained by (14). Otherwise, (S_2, T_2) is infeasible and set $AC_2(S_2, T_2) = \infty$.

Step 3. Comparing $AC_1(S_1, T_1)$ with $AC_2(S_2, T_2)$, we obtain $AC(S^*, T^*) = \mathrm{Min}\{AC_1(S_1, T_1), AC_2(S_2, T_2)\}$. The optimal order quantity is $Q^* = I_2(S^*) - I_1(S^*)$, and stop.

Case 3. The retailer pays off the total amount on its account at time $M+S$ with $S < M+S \leqslant T$.

To minimize the total annual relevant cost, taking the first derivative of $AC_3\,(S, T)$ with respect to S and T, and setting the results to be zero, we obtain

$$\frac{\partial AC_3(S,T)}{\partial T} = \frac{-AC_3\,(S,T)}{T} + \frac{D}{T}\Bigg[\mathrm{e}^{\theta(T-S)}\left(\frac{C_h}{\theta} + C_p + I_e\,(T-M-S)\,C_p\right) - \frac{C_h}{\theta} - I_e f\,(S,T)\,/D - I_e p\,(T-M-S)\Bigg] = 0, \tag{30}$$

and

$$\begin{aligned}\frac{\partial AC_3(S,T)}{\partial S} = \frac{D}{T}\Bigg\{ & -\frac{C_0 I_e}{D}+\frac{C_h}{\theta}\left[1-e^{\theta(T-S)}\right]+C_s S\beta(S)+C_l[1-\beta(S)] \\ & +C_p\left[-e^{\theta(T-S)}+\beta(S)\right]-pI_e M\beta(S) \\ & +I_e\int_0^S \beta(t)dt\left(p+pI_e-C_p\right)+pM\left(1+I_e\frac{M}{2}\right) \\ & -C_p\left[e^{\theta(T-S)}-1\right]-I_e(T-M-S) \\ & \times\left(\beta(S)(p+pI_eM-C_h)+C_p e^{\theta(T-S)}-p\right)\Bigg\}=0. \qquad (31)\end{aligned}$$

Let S_3^* and T_3^* denote the optimal values of S and T, respectively, then the optimal solution (S_3^*, T_3^*) satisfies (30) and (31) simultaneously. Solving (30) and (31) by Newton–Raphson method (or any other bisection method), we can obtain the values of S_3^* and T_3^*. If the Hessian matrix of $AC_3(S,T)$ is positive definition, then the optimal solution (S_3^*, T_3^*) of $AC_3(S,T)$ is global minimum. Similar to the above Algorithm, the reader can develop an algorithm to determine the optimal solution for Cases 3 and 4.

5. Numerical Examples

In order to illustrate the preceding results, we provide the following numerical examples. We study the differences between Abad's exponential backlogging rate and reciprocal backlogging rate.

EXAMPLE 1. Given $\theta = 0.08$, $D = 1000$ units/year, $C_0 = \$250$ per order, $C_h = \$80$/unit, $C_s = \$120$/unit, $C_l = \$300$/unit, $C_p = \$150$/unit, $p = \$240$/unit, $I_c = 0.06$/year, $I_e = 0.04$/year, $M = \{15/365, 30/365, 45/365 \text{ or } 60/365\}$ years and $\beta(x) = e^{-\alpha x}$ (e.g., see Papachristos and Skouri, 2000), where $\alpha = \{0, 0.6, 1, 5, 10, 20, \text{ or } 50\}$. The optimal values of $AC(S,T)$ for different values of α, M and $\beta(x) = e^{-\alpha x}$ are shown in Table 1. Note that the unit of S^* and T^* is in years in all tables below.

EXAMPLE 2. In this example, we use the same parameters as in Example 1. However, we adopt the backlogging rate $\beta(x) = (1+\alpha x)^{-1}$ as in Chang and Dye (1999). The computational results are shown in Table 2.

Comparing the total annual relevant costs between Tables 1 and 2, we know that there is no significant difference between the negative exponential backlogging rate as in Papachristos and Skouri (2000) and the inverse linear backlogging rate as in Chang and Dye (1999). Consequently, for convenience, we will use the negative exponential backlogging rate to do the

Table 1. The optimal solutions of $AC(S,T)$ with $\beta(x)=e^{-\alpha x}$

		α						
M		0	0.6	1	5	10	20	50
15/365(=0.04110)	S^*	0.04376	0.02821	0.02284	0.00791	0.00436	0.00230	0.00095
	T^*	0.09532	0.08569	0.08254	0.07425	0.07239	0.07133	0.07064
	Q^*	95.4279	85.5812	82.4200	74.2730	72.4840	71.4707	70.8149
	$AC(S^*,T^*)$	154847	155448	155672	156347	156518	156620	156688
30/365(=0.08219)	S^*	0.04377	0.02818	0.02282	0.00790	0.00436	0.00230	0.00095
	T^*	0.09527	0.08559	0.08243	0.07412	0.07226	0.07120	0.07051
	Q^*	95.3726	85.4832	82.3180	74.1455	72.3534	71.3386	70.6819
	$AC(S^*,T^*)$	154453	155055	155280	155956	156127	156229	156297
45/365(=0.12329)	S^*	0.04377	0.02816	0.02277	0.00788	0.00434	0.00229	0.00095
	T^*	0.09527	0.08558	0.08241	0.07411	0.07226	0.07120	0.07051
	Q^*	95.3726	85.4675	82.2954	74.1455	72.3483	71.3358	70.6807
	$AC(S^*,T^*)$	154059	154662	154887	155562	155733	155835	155902
60/365(=0.16438)	S^*	0.04377	0.02813	0.02276	0.00786	0.00433	0.00228	0.00094
	T^*	0.09527	0.08556	0.08239	0.07411	0.07217	0.07119	0.07051
	Q^*	95.3726	85.4518	82.2719	74.1283	72.3433	71.3333	70.6795
	$AC(S^*,T^*)$	153664	154268	154494	155168	155339	155441	155508

Table 2. The optimal solutions of $AC(S,T)$ with $\beta(x)=(1+\alpha x)^{-1}$

		α						
M		0	0.6	1	5	10	20	50
15/365(=0.04110)	S^*	0.04376	0.02829	0.02297	0.00804	0.00445	0.00235	0.00097
	T^*	0.09532	0.08575	0.08263	0.07434	0.07245	0.07137	0.07066
	Q^*	95.4279	85.6479	82.5144	74.3611	72.5415	71.5038	70.8317
	$AC(S^*,T^*)$	154847	155446	155669	156343	156516	156619	156689
30/365(=0.08219)	S^*	0.04377	0.02827	0.02295	0.00803	0.00444	0.00235	0.00097
	T^*	0.09527	0.08566	0.08252	0.07421	0.07232	0.07123	0.07053
	Q^*	95.3726	85.5499	82.4062	74.2334	72.4108	71.3715	70.6986
	$AC(S^*,T^*)$	154453	155053	155277	155951	156125	156228	156296
45/365(=0.12329)	S^*	0.04377	0.02825	0.02292	0.00801	0.00443	0.00234	0.00097
	T^*	0.09527	0.08564	0.08251	0.07421	0.07232	0.07123	0.07053
	Q^*	95.3726	85.5342	82.3897	74.2245	72.4055	71.3687	70.6973
	$AC(S^*,T^*)$	154059	154660	154883	155558	155731	155833	155901
60/365(=0.16438)	S^*	0.04378	0.02822	0.02289	0.00799	0.00442	0.00233	0.00097
	T^*	0.09526	0.08562	0.08249	0.07420	0.07231	0.07123	0.07052
	Q^*	95.3726	85.5185	82.3732	74.2156	72.4003	71.3658	70.6961
	$AC(S^*,T^*)$	153664	154266	154490	155164	155337	155439	155507

sensitivity analysis for Cases 1 and 2 in different values of α, θ, I_c, I_e, and C_1 in Examples 3–5.

EXAMPLE 3. The parameters are the same as in Example 1, except for $M=30/365$ years. However, we use different values of θ to study its effects on the optimal solution. The computational results are shown in Table 3.

Table 3. The sensitivity analysis on different parameters α and θ

		α						
θ		0	0.6	1	5	10	20	50
0.05	S^*	0.04321	0.02775	0.02244	0.00773	0.00426	0.00224	0.33810
	T^*	0.09647	0.08693	0.08382	0.07568	0.07386	0.07282	0.07215
	Q^*	96.5386	86.7905	83.6681	75.6459	73.8896	72.8958	72.2519
	$AC(S^*, T^*)$	154388	154966	155180	155818	155980	156075	156138
0.10	S^*	0.04411	0.02849	0.02307	0.00801	0.00442	0.00233	0.00096
	T^*	0.09451	0.08475	0.08155	0.07315	0.07126	0.07018	0.06948
	Q^*	94.6375	84.6634	81.4603	73.2006	71.3853	70.3567	69.6916
	$AC(S^*, T^*)$	154495	155113	155345	156045	156224	156330	156400

Table 4. The sensitivity analysis on different parameters I_c and I_e

		I_c				
I_e		0.02	0.04	0.06	0.08	0.10
0.02	S^*	0.02243	0.02246	0.02248	0.02251	0.02253
	T^*	0.08422	0.08370	0.08320	0.08272	0.08228
	Q^*	84.1267	83.5963	83.0943	82.6183	82.1664
	$AC(S^*, T^*)$	155761	155769	155776	155783	155789
0.08	S^*	0.02350	0.02352	0.02354	0.02356	0.02358
	T^*	0.08203	0.08157	0.08114	0.08073	0.08035
	Q^*	81.8920	81.4335	80.9998	80.5889	80.1990
	$AC(S^*, T^*)$	155449	155455	155460	155465	155469

EXAMPLE 4. In this example, we investigate the effects of I_c and I_e on the total annual relevant cost when $M = 15/365$ years, $\theta = 0.08$ and $\alpha = 1$. The other parameters are the same as in Example 1. Consequently, we obtain the numerical results as shown in Table 4.

EXAMPLE 5. In this example, we study the effect of C_l if $\alpha = 0$, 5, 10 or 50, $I_c = 0.08$/year, and $I_e = 0.02$/year. The rest of the parameters are the same as in Example 1. Consequently, we obtain the numerical results as shown in Table 5.

EXAMPLE 6. For illustrating the retailer pays off the total amount on its account at time $M + S$ (i.e., Cases 3 and 4), we give the following data: $\theta = 0.08$, $D = 1000$ units/year, $C_0 = \{\$250, \text{or}\$1000\}$ per order, $C_h = \$80$/unit, $C_s = \$120$/unit, $C_l = \$300$/unit, $C_p = \$150$/unit, $p = \{\$240, \text{or } \$300\}$/unit, $I_c = 0.06$/year, $I_e = 0.04$/year, $M = \{15/365, 30/365, \text{ or } 45/365\,\text{years}\}$ and $\beta(x) = e^{-\alpha x}$, where $\alpha = \{1, \text{or } 10\}$. The numerical results for different values of p, C_0, α, M and $\beta(x) = e^{-\alpha x}$ are shown in Table 6.

Table 5. The sensitivity analysis on different values of C_1 and α

		$C_1=200$				$C_1=450$			
		α				α			
M		0	5	10	50	0	5	10	50
30/365(=0.08219)	S^*	0.04315	0.01760	0.01105	0.00278	0.04315	0.00422	0.00222	0.00046
	T^*	0.09652	0.08124	0.07760	0.07317	0.09652	0.07389	0.07286	0.07197
	Q^*	96.6349	80.6533	77.1895	73.1868	96.6439	74.0405	73.0405	72.1727
	$AC\,(S^*, T^*)$	154781	155781	156063	156438	154781	156368	156462	156546
45/365(=0.12329)	S^*	0.04315	0.01756	0.01102	0.00277	0.04315	0.00422	0.00222	0.00046
	T^*	0.09652	0.08122	0.07758	0.07317	0.09652	0.07389	0.07286	0.07197
	Q^*	96.6349	80.6313	77.1377	73.1823	96.6439	74.0392	73.0398	72.1727
	$AC\,(S^*, T^*)$	154584	155585	155868	156241	154784	156171	156265	156349

Table 6. The optimal solutions of $AC\,(S, T)$ with $\beta(x)=e^{-\alpha x}$ for Cases 3 and 4

		$C_0=250$				$C_0=1000$			
		$p=240$		$p=300$		$p=240$		$p=300$	
		α		α		α		α	
M		1	10	1	10	1	10	1	10
15/365(=0.04110)	S^*	0.02285	0.00437	0.02309	0.04551	0.04650	0.00895	0.04693	0.00905
	T^*	0.08552	0.07494	0.08694	0.07596	0.17130	0.14971	0.17410	0.15173
	Q^*	85.418	75.049	86.834	76.074	170.856	150.117	173.667	152.152
	$AC\,(S^*, T^*)$	155602	156421	155506	156331	161436	163095	161246	162915
30/365(=0.08219)	S^*	0.02282	0.00436	0.02300	0.00440	0.04646	0.00894	0.04689	0.00903
	T^*	0.08243	0.07226	0.08174	0.07148	0.17130	0.14973	0.17411	0.15175
	Q^*	82.3128	72.355	81.611	71.563	170.864	150.134	173.679	152.172
	$AC\,(S^*, T^*)$	155280	156127	155135	156006	161190	162847	160999	162668
45/365(=0.12329)	S^*	0.02279	0.00434	0.02297	0.00439	0.04637	0.00892	0.04673	0.00901
	T^*	0.08241	0.07226	0.08171	0.07147	0.16518	0.14975	0.16380	0.15178
	Q^*	82.296	72.350	81.5908	71.556	164.684	150.163	163.273	152.207
	$AC\,(S^*, T^*)$	154887	155733	154643	155514	160939	162600	160747	162421

Based on the computational results as shown in Tables 1–5, we obtain the following managerial phenomena:

(1) For fixed M and θ, if the value of α is increasing, then the optimal length of the shortage period S^*, the optimal length of the cycle period T^* and the optimal order quantity Q^* are decreasing simultaneously, yet the optimal total annual relevant cost $AC(S^*, T^*)$ is increasing. A simple economic interpretation is as follows. A larger value of α indicates a smaller backlogging rate. Therefore, the shortage period S^* and the order quantity Q^* are getting smaller. However, the total cost $AC(S^*, T^*)$ is getting larger because a smaller backlogging rate implies a larger amount of lost sales.
(2) For fixed $\alpha > 0$ and θ, if the value of M is increasing, then the optimal length of the shortage period S^*, the optimal length of the cycle period T^*, the optimal order quantity Q^* and the optimal total annual relevant cost $AC(S^*, T^*)$ are decreasing simultaneously. A simple economic interpretation is as follows. If the value of M is increasing, then the benefit of the permissible delay is increasing. As a result, the retailer should order less quantity Q^* and shorten the cycle T^* in order to take the permissible delay more often.
(3) For fixed M and α, if the value of θ is increasing, then the optimal length of the shortage period S^* and the optimal total annual relevant cost$AC(S^*, T^*)$ are increasing simultaneously, yet the optimal length of the cycle period T^* and the optimal order quantity Q^* are decreasing simultaneously. A simple economic interpretation is as follows. If the deterioration rate θ is high, then the retailer should order small quantity Q^* (which implies a small cycle time T^*) and increase shortage period S^* in order to reduce deteriorated items.
(4) For fixed M, θ, α and I_e, if the value of I_c is increasing, then the optimal length of the shortage period S^* and the optimal total annual relevant cost $AC(S^*, T^*)$ are increasing simultaneously, yet optimal length of the cycle period T^* and the optimal order quantity Q^* are decreasing simultaneously. A simple economic interpretation is as follows. If the interest rate charged is higher, then the retailer should order less quantity Q^* (so is a smaller T^*) and make the shortage period S^* longer in order to reduce the interest charged by the supplier.
(5) For fixed M, θ, α and I_c, if the value of I_e is increasing, then the optimal length of the cycle period T^*, the optimal order quantity Q^* and the optimal total annual relevant cost $AC(S^*, T^*)$ are decreasing simultaneously, yet the optimal length of the shortage period S^* is increasing. A simple economic interpretation is as follows. If the interest rate earned I_e is increasing, then the benefit of the permissible delay is increasing, too. As a result, the retailer should order less

quantity Q^* and shorten the cycle T^* in order to take the permissible delay more frequently.

(6) For fixed M and $\alpha > 0$, if the value of C_l is increasing, then the optimal length of the shortage period S^*, the optimal length of the cycle period T^* and the optimal order quantity Q^* are decreasing simultaneously, yet the optimal total annual relevant cost $AC(S^*, T^*)$ is increasing. A simple economic interpretation is as follows. If cost of lost sales C_l is high, then the retailer should shorten the shortage period S^* to reduce the number of lost sales.

Note that our second computational result contradicts the previous conclusion that the replenishment cycle generally increases, such as in Goyal (1985), Jamal et al. (1997) and others. Our interpretation to this conflict is the same as in Teng (2002). If a supplier provides a longer permissible delay period to his/her retailer, then the retailer receives larger amount of interest earned whenever he/she places an order. As a result, the retailer should order less quantity and more often (i.e., a shorter replenishment cycle time) in order to take the benefits of trade credit more frequently.

6. Conclusions

In this paper, we first established an appropriate mathematical model for deteriorating items when the supplier offers a permissible delay in payment. Our model is in a general framework that includes numerous previous models as special cases. We then provided the theoretical results to show that the total annual relevant cost is a strictly pseudo-convex function. As a result, we proved that there exists a unique interior optimal solution to the proposed model, which simplifies the search for the global minimum to finding a local minimum. Furthermore, we studied the sensitivity analysis on the parameters, and concluded some interesting managerial phenomena, such as the larger the α (or M), the smaller the S^* (and Q^*).

The proposed model can be extended in several ways. For instance, we may extend the deterministic demand function to a stochastic demand pattern. Also, we could generalize the model to allow for quantity discount. Finally, we could consider the problem of simultaneously setting price, quality, and order quantity for a product in which its demand is a function of unit selling price as well as product quality.

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Appendix A. Proof that Hessian matrix of $AC_1(S,T)$ at point (S_1^*, T_1^*) is positive definite

$$\frac{\partial^2 \mathrm{AC}_1(S,T)}{\partial T^2} = \frac{D}{T} e^{\theta(T-S)}\left(C_h + \theta C_p + C_p I_c e^{-\theta M}\right) - \frac{2}{T}\left(\frac{\partial \mathrm{AC}_1(S,T)}{\partial T}\right), \tag{A.1}$$

$$\frac{\partial^2 \mathrm{AC}_1(S,T)}{\partial S^2} = \frac{D}{T}\left\{e^{\theta(T-S)}\left(C_h + \theta C_p + C_p I_c e^{-\theta M}\right) - \left(C_l - C_p\right)\beta'(S) + C_s\left[\beta(S) + S\beta'(S)\right] - pI_e M\beta'(S)\right\}, \tag{A.2}$$

and

$$\frac{\partial^2 \mathrm{AC}_1(S,T)}{\partial S \partial T} = -\frac{1}{T}\left(\frac{\partial \mathrm{AC}_1(S,T)}{\partial S}\right) - \frac{D}{T} e^{\theta(T-S)}(C_h + \theta C_p + C_p I_c e^{-\theta M}). \tag{A.3}$$

The first principal minor of the Hessian matrix $\mathbf{H}$ at point (S_1^*, T_1^*) is

$$|\mathbf{H}_{11}| = \left.\frac{\partial^2 \mathrm{AC}_1(S,T)}{\partial T^2}\right|_{S_1^*, T_1^*} = \frac{D}{T_1^*} e^{\theta(T_1^* - S_1^*)}(C_h + \theta C_p + C_p I_c e^{-\theta M}) > 0. \tag{A.4}$$

The second principal minor of $\mathbf{H}$ at point $\left(S_1^*, T_1^*\right)$ is

$$\begin{aligned}|\mathbf{H}_{22}| &= \left(\frac{D}{T_1^*}\right)^2 e^{\theta(T_1^* - S_1^*)}\left(C_h + \theta C_p + C_p I_c e^{-\theta M}\right)\\ &\quad\times\{-\left(C_l - C_p\right)\beta'\left(S_1^*\right) + C_S[\beta\left(S_1^*\right) + S_1^*\beta'\left(S_1^*\right)] - pI_e M\beta'\left(S_1^*\right)\}\\ &= \left(\frac{D}{T_1^*}\right)^2 e^{\theta(T_1^* - S_1^*)}(C_h + \theta C_p + C_p I_c e^{-\theta M})\\ &\quad\times\{[(C_p + C_s S_1^*) - (C_l + pI_e M)]\beta'(S_1^*) + C_S\beta(S_1^*)\}.\end{aligned} \tag{A.5}$$

Note that one of the referees provided us a brilliant and simple proof of $|\mathbf{H}_{22}| > 0$ as follows. The maximal total cost (i.e., purchase cost and shortage cost, or $C_p + C_s S$) of satisfying a unit of backlogged demand must less than the total opportunity cost (i.e., cost of lost sales and interest that could be earned, or $C_1 + pI_eM$) to turn down a unit of demand. Otherwise, the retailer will just ignore the demand backlogged at the beginning of the cycle since it is not profitable to satisfy the backlogged demand. As a result of $C_p + C_sS < C_1 + pI_eM$, we can easily obtain that $|H_{22}| > 0$. This completes the proof.

Appendix B. Proof of Theorem 1

Let $F(S,T) = \mathrm{AC}_1(S,T)T > 0$. Similar to the proof in Appendix A, we can easily show that $F(S,T)$ is a strictly convex function for $0 \leqslant S < T$. Next, for any two distinct points in J, say (S_1, T_1) and (S_2, T_2), if $\mathrm{AC}_1\,(S_1, T_1) \geqslant \mathrm{AC}_1\,(S_2, T_2)$, then

$$\frac{F\,(S_2,T_2)}{F\,(S_1,T_1)} \leqslant \frac{T_2}{T_1} \Leftrightarrow \frac{F\,(S_2,T_2) - F\,(S_1,T_1)}{F\,(S_1,T_1)} \leqslant \frac{T_2 - T_1}{T_1}. \tag{B.1}$$

Since $F(S,T)$ is a strictly convex function, we know

$$F(S_2,T_2) > F(S_1,T_1) + \left.\frac{\partial F(S,T)}{\partial T}\right|_{S_1,T_1} (T_2 - T_1) + \left.\frac{\partial F(S,T)}{\partial S}\right|_{S_1,T_1} (S_2 - S_1), \tag{B.2}$$

and hence

$$F(S_2,T_2) - F(S_1,T_1) > \left.\frac{\partial F(S,T)}{\partial T}\right|_{S_1,T_1} (T_2 - T_1) + \left.\frac{\partial F(S,T)}{\partial S}\right|_{S_1,T_1} (S_2 - S_1). \tag{B.3}$$

Combining (B1) and (B3), we obtain

$$\left.\frac{\partial F\,(S,T)}{\partial T}\right|_{S_1,T_1} (T_2 - T_1) + \left.\frac{\partial F\,(S,T)}{\partial S}\right|_{S_1,T_1} (S_2 - S_1) < \frac{F\,(S_1,T_1)\,[T_2 - T_1]}{T_1}, \tag{B.4}$$

Dividing (B4) by T_1, we have

$$\frac{\left.\frac{\partial F(S,T)}{\partial T}\right|_{S_1,T_1} (T_2 - T_1) + \left.\frac{\partial F(S,T)}{\partial S}\right|_{S_1,T_1} (S_2 - S_1)}{T_1} < \frac{\mathrm{AC}_1(S_1,T_1)\,[T_2 - T_1]}{T_1},$$

$$\Leftrightarrow \frac{\left[\left.\frac{\partial F(S,T)}{\partial T}\right|_{S_1,T_1} - \mathrm{AC}_1(S_1,T_1)\right](T_2 - T_1) + \left.\frac{\partial F(S,T)}{\partial S}\right|_{S_1,T_1} (S_2 - S_1)}{T_1} < 0$$

$$\Leftrightarrow \left.\left[\frac{1}{T}\frac{\partial F(S,T)}{\partial T} - \frac{F(S,T)}{T^2}\right]\right|_{S_1,T_1} (T_2 - T_1) + \left.\frac{1}{T}\frac{\partial F(S,T)}{\partial S}\right|_{S_1,T_1} (S_2 - S_1) < 0$$

$$\Leftrightarrow \left.\frac{\partial \mathrm{AC}_1(S,T)}{\partial T}\right|_{S_1,T_1}(T_2-T_1)+\left.\frac{\partial \mathrm{AC}_1(S,T)}{\partial S}\right|_{S_1,T_1}(S_2-S_1)<0. \tag{B.5}$$

Applying the result obtained by Bazarra et al. (1993, p. 114), we know that $\mathrm{AC}_1(S,T)$ is a strictly pseudo-convex function on J. This completes the proof of (a).

From (16), we set

$$\begin{aligned} G(S)=\frac{D}{T}\bigg\{&-\frac{C_0 I_\mathrm{e}}{D}+\frac{C_\mathrm{h}}{\theta}\left[1-\mathrm{e}^{\theta(T-S)}\right]+C_\mathrm{s}S\beta(S)+C_\mathrm{l}[1-\beta(S)]\\ &+C_\mathrm{p}\left[-\mathrm{e}^{\theta(T-S)}+\beta(S)\right]+\frac{C_\mathrm{p}I_\mathrm{c}}{\theta}\left[1-\mathrm{e}^{\theta(T-M-S)}\right]-pI_\mathrm{e}M\beta(S)\bigg\}. \end{aligned} \tag{B.6}$$

Since $\mathrm{AC}_1\,(S,T)$ is strictly convex in S, we know that the first derivate of $G(S)$ with respect to S must be positive (i.e., $G'(S)>0$). Hence, $G(S)$ is a strictly increasing function in S, for $S\in(0,T)$. In addition, we have

$$\begin{aligned} G(0)=\frac{D}{T}\bigg\{&-\frac{C_0 I_\mathrm{e}}{D}-\frac{C_\mathrm{h}}{\theta}\left[\mathrm{e}^{\theta T}-1\right]-C_\mathrm{p}\left[\mathrm{e}^{\theta T}-1\right]\\ &-\frac{C_\mathrm{p}I_c}{\theta}\left[\mathrm{e}^{\theta(T-M)}-1\right]-pI_\mathrm{e}M\bigg\}<0. \end{aligned} \tag{B.7}$$

$$\begin{aligned} G(T)=\frac{D}{T}\bigg\{&-\frac{C_0 I_\mathrm{e}}{D}-pI_\mathrm{e}M\beta(T)+C_\mathrm{s}T\beta(T)+\left(C_\mathrm{l}-C_\mathrm{p}\right)[1-\beta(T)]\\ &+\frac{C_\mathrm{p}I_c}{\theta}\left[1-\mathrm{e}^{-\theta M}\right]\bigg\}. \end{aligned} \tag{B.8}$$

Hence, if $C_\mathrm{s}T\beta(T)+(C_\mathrm{l}-C_\mathrm{p})[1-\beta(T)]+\frac{C_\mathrm{p}I_c}{\theta}[1-\mathrm{e}^{-\theta M}]>\frac{C_0 I_\mathrm{e}}{D}+pI_\mathrm{e}M\beta(T)$, then $G(T)>0$, which in turn implies that there exists a unique $S^*\in(0,T)$ such that $G(S^*)=0$. This completes the proof of (b).

Similarly, if $C_\mathrm{s}T\beta(T)+(C_\mathrm{l}-C_\mathrm{p})[1-\beta(T)]+\frac{C_\mathrm{p}I_c}{\theta}[1-\mathrm{e}^{-\theta M}]\leqslant\frac{C_0 I_\mathrm{e}}{D}+pI_\mathrm{e}M\beta(T)$, then $G(T)\leqslant 0$, which implies that $\mathrm{AC}_1\,(S,T)$ is minimized if $S_1^*=T_1$. This proves (c).

Now, let us prove part (d). If $T\to 0^+$, then $S\to 0^+$, $M\to 0^+$, and thus

$$\lim_{T\to 0^+}\mathrm{AC}_1\,(S,T)=\lim_{T\to 0^+}C_0/T=\infty. \tag{B.9}$$

Similarly, if $T \to \infty$, then

$$\lim_{T\to\infty} \mathrm{AC}_1\,(S,T) = \left\{ \lim_{T\to\infty} \frac{H'(T)}{1} \right\}$$

$$- \frac{C_{\mathrm{h}}D}{\theta} - \frac{C_{\mathrm{p}}DI_{\mathrm{c}}}{\theta} \text{(by L'Hôpital's Rule)},$$

$$= \lim_{T\to\infty} \left\{ \frac{C_{\mathrm{h}}De^{\theta(T-S)}}{\theta} + C_{\mathrm{p}}De^{\theta(T-S)} + \frac{C_{\mathrm{p}}DI_{\mathrm{c}}e^{\theta(T-M-S)}}{\theta} \right\}$$

$$- \frac{C_hD}{\theta} - \frac{C_{\mathrm{p}}DI_{\mathrm{c}}}{\theta} = \infty, \tag{B.10}$$

where

$$H(T) = \left\{ C_0(1 - SI_{\mathrm{e}}) + \frac{C_{\mathrm{h}}D\left[e^{\theta(T-S)} - 1 + \theta S\right]}{\theta^2} + C_S D \int_0^S t\beta(t)\mathrm{d}t \right.$$

$$+ C_1 D\left[S - \int_0^S \beta(t)\mathrm{d}t\right] + \frac{C_{\mathrm{p}}D\left[\left(e^{\theta(T-S)} - 1\right) + \theta \int_0^S \beta(t)\mathrm{d}t\right]}{\theta}$$

$$+ \frac{C_{\mathrm{p}}DI_{\mathrm{c}}\left[e^{\theta(T-M-S)} - 1 + \theta\,(M+S)\right]}{\theta^2}$$

$$\left. - \frac{pDI_{\mathrm{e}}M\left[M + 2\int_0^S \beta(t)\mathrm{d}t\right]}{2} \right\}. \tag{B.11}$$

From (a), (B9), and (B10), we know there exists a unique optimal T_1^* (with $0 < T_1^* < \infty$). This completes the proof of (d).

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